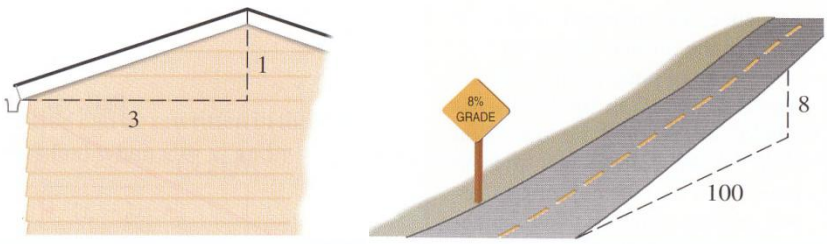
§ 1.10: Lines

The Slope of a Line

For a line, we will define the *run* to be the distance we move to the right and the *rise* to be the corresponding distance that the rises (or falls). The *slope* of the line is the ratio of rise to run:

$$slope = \frac{rise}{run}$$

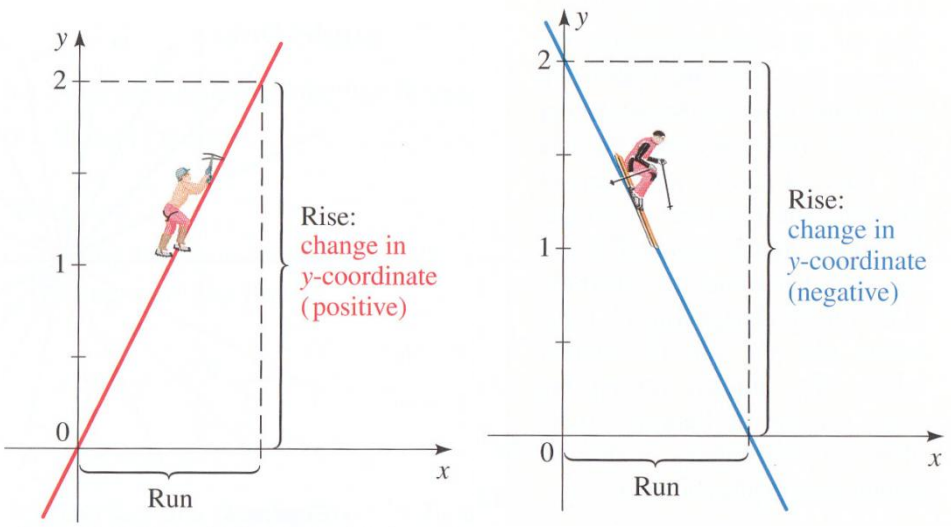
The following pictures illustrate some applications of slope.



Pitch of a roof
Slope $= \frac{1}{3}$

Grade of a road
Slope $= \frac{8}{100}$

If a line lies in the xy-plane, then the rise is the change in the y-coordinate while the run is the change in the x-coordinate.



This gives the following definition of slope.

Slope of a line

The **slope** of a non-vertical line that passes through the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of a vertical line is undefined.

Example 1	Finding the Slope of a Line through Two Points

Find the slope of the line that passes through the points (2,1) and (8,8).

Equations of Lines

Now let's find an equation of the line passing through a given point $A(x_1,y_1)$ and has slope *m*. A point B(x,y), with $x \neq x_1$, lies on this line if and only if the slope of the line through A and B is equal to *m*, that is

$$\frac{y - y_1}{x - x_1} = m.$$

By multiplying both sides by $x-x_1$ we obtain

Point-Slope Form of the Equation of a Line

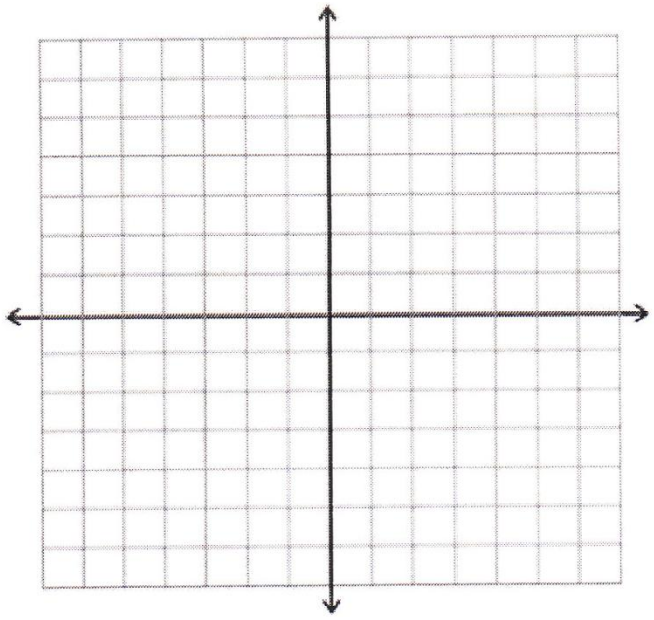
An equation of the line that passes through the point (x_1,y_1) and has slope *m* is

$$y - y_1 = m(x - x_1).$$

Example 2	Finding the Equation of a Line with Given Point and Slope

(a) Find an equation of the line which passes through the point (-1, -4) with slope $-\frac{1}{3}$.

(b) Sketch the graph of the line



Example 3	Finding the Equation of a Line through Two Given Points

Find an equation of a line through the points (-3,5) and (2,-7).

Now suppose a non-vertical line has slope m and passes through the point (0,b), here we call b the **y-intercept** of the line. We can therefore write out the point-slope form of this line, it is

$$y - b = m(x - 0).$$

Solving this equation for y yields:

$$y = mx + b.$$

Slope-Intercept Form of the Equation of a Line

An equation of the line that has slope m and y-intercept b is given by

$$y = mx + b.$$

Example 4	Lines in Slope-Intercept Form

(a) Find the equation of the line with slope 5 and y-intercept -10.

(b) Find the slope and y-intercept of the line $4y - 8x + 1 = 0$.

If a line is horizontal, then its slope is $m = 0$; if it is vertical its slope is undefined. However, we can still write an equation for a vertical line.

Vertical and Horizontal Lines

An equation of the vertical line passing through the point (a,b) is x = 1.

An equation of the horizontal line passing through the point (a,b) is y = b.

General Equation of a Line

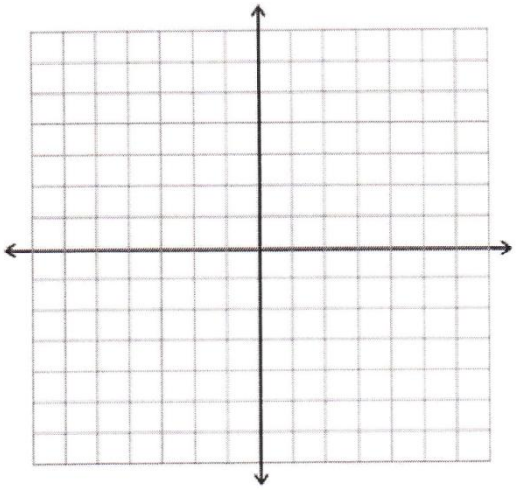
The graph of every **linear equation**

$$Ax + By + C = 0 \quad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

Example 5	Graphing a Linear Equation

Sketch the graph of the equation $3x - 2y - 12 = 0$.



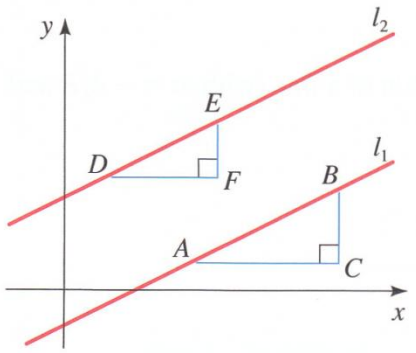
Parallel and Perpendicular Lines

Parallel Lines

Two non-vertical lines are parallel if and only if they have the same slope.
Any two vertical lines are parallel.

proof:

Let the lines l_1 and l_2 in the following figure have slopes m_1 and m_2.



If the lines are parallel, then the right triangles ABC and DEF are similar, so that

$$m_1 = \frac{d(B,C)}{d(A,C)} = \frac{d(E,F)}{d(D,F)} = m_2.$$

Conversely, if the slopes are equal, then the triangles will be similar, so ∠BAC = ∠EDF; therefore the lines are parallel.

Example 6	Finding the Equation of a Line Parallel to a Given Line

Find an equation of the line passing through the point (5,2) that is parallel to the line 2x + 6y + 12 = 0.

Perpendicular Lines

Two limes with slopes m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}.$$

Also, a horizontal line is perpendicular to a vertical line.

Proof:

Suppose lines l_1 and l_2 intersect at the origin, O; furthermore, suppose they are perpendicular.

If we let their slopes be m_1 and m_2, then there equations would be $y = m_1x$ and $y = m_2x$. Notice that the point $A(1,m_1)$ is on l_1 and the point $B(1,m_2)$ is on l_2. By the Pythagorean Theorem and its converse, $OA \perp OB$ if and only if

$$[d(O,A)]^2 + [d(O,B)]^2 = [d(A,B)]^2.$$

This, by the distance formula becomes

$$\begin{aligned} (1 + {m_1}^2) + (1 + {m_2}^2) &= (1-1)^2 + (m_2 - m_1)^2 \\ 2 + {m_1}^2 + {m_2}^2 &= {m_2}^2 - 2m_2m_1 + {m_1}^2 \\ 2 &= -2m_2m_1 \\ m_1m_2 &= -1 \end{aligned}$$

Example 7	Perpendicular Lines

Show that the points $P(3,3)$, $Q(8,17)$, and $R(11,5)$ are the vertices of a right triangle.

Example 9	Finding an Equation of a Line Perpendicular to a Given Line

Find an equation of the line that is perpendicular to the line $4x + 2y + 7 = 0$ and passes through the origin.

Applications: Slope as Rate of Change

Example 10	Slope as Rate of Change

A dam is built on a river to create a reservoir. The water level w in the reservoir is given by the equation

$$w = 4.5t + 28$$

where t is the number of years since the dam was constructed, and w is measured in feet.

What do the slope and w-intercept of this line represent?

Example 11	Linear Relationship between Temperature and Elevation

(a) As dry air moves upward, it expands and cool. If the gound temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) in terms of height h (in kilometers). (Assume that the relationship between T and h is linear.)

(b) What is the temperature at a height of 2.5 km?

Homework

Due:_____

2 – 36 (even), 42 – 52 (even), 54, 60, 68